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Mass spectra and leptonic decay widths of heavy quarkonia*

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Abstract. A nonrelativistic Hamiltonian with plausible spin dependent corrections is proposed for the quarkonia below their respective strong decay thresholds. With only six free parameters this model reproduces the nine known masses of the bottomonia within about 1 MeV, the six known masses of the charmonia within a few MeV and the five known leptonic decay widths of the $^3S_1$ states within about 20 %. The model is then used to predict the masses of the remaining 43 quarkonia (some of them for the first time) and of the leptonic decay widths of the two $^1S_0(\bar{b}c)$ states. Comparison with some other models is made.

1 Introduction

In the present paper we study the mass spectra of heavy quarkonia below their strong decay thresholds, i.e. below 10558 MeV for the $\bar{b}b$ quarkonia, below 7144 MeV for the $\bar{b}c$ quarkonia and below 3729 MeV for the $\bar{c}c$ quarkonia. For the quarkonia with unnatural parity, which because of the conservation laws cannot decay strongly into two pseudoscalar mesons, these thresholds should be a little higher. In practice this distinction is important only for the quarkonia $2P_1(\bar{b}c)$ and $2P_{1'}(\bar{b}c)$, which according to our calculation have masses above the minimal thresholds given above, but below their real threshold, which is 7189 MeV. As a byproduct we obtain the leptonic widths of the $^3S_1$ states of the $\bar{b}b$ and of the $\bar{c}c$ quarkonia, as well as of the $^1S_0$ states of the $\bar{b}c$ quarkonia. Experimentally, out of the 34 $\bar{b}b$ quarkonia expected nine have been observed. Here and in the following we consider a particle as observed, if it is listed as firmly established in the 1996 Particle Data Group Tables [1]. Out of the eight expected $\bar{c}c$ quarkonia six have been observed and only the singlet P state and the excited η_c are still missing. The masses of both have been reported [1], but they are not considered as firmly established. None of the $\bar{b}c$ quarkonia has been observed, but candidates have been reported [2] and discoveries are expected in the near future.

In our previous paper [3] (further quoted I) we have pointed out that a simple nonrelativistic model can reproduce, among other things, the masses of the known $^3S(\bar{b}b)$ states and of the centres of gravity of the known $^3P(\bar{b}b)$ states within the experimental errors. After this paper had been published, we became aware of a series of papers (cf. [4] and papers quoted there) using essentially the same potential. There are many models, which give good fits to the masses of the $\bar{b}b$ quarkonia. The review paper [5] quotes and discusses about 30 of them. Our model, however, seems to be the only one so far, which reproduces these masses within the experimental errors, i.e. with a precision of about 0.5 MeV, which corresponds to 50 ppm. of the total mass, or to 0.1 per cent of the first excitation energy. This result is amazing, because the mean square velocity of a b-quark in the ground state of the $\bar{b}b$ system is, in a system of units where the velocity of light $c = 1$, $\langle v^2 \rangle \approx 0.08$. Thus a simple-minded estimate of the relativistic corrections would give $\langle v^2 \rangle^2 \approx 0.6\%$ of the total mass. The usual interpretation is that the nonrelativistic Hamiltonian is an effective Hamiltonian and that the relativistic corrections are taken into account by a renormalization of the parameters of this Hamiltonian.

In the present paper we extend the model described in I in two ways. Firstly, we generalize the nonrelativistic potential so that it applies also to $\bar{b}c$ and $\bar{c}c$ quarkonia. This requires one more parameter — the mass of the c-quark. We find that the predictions for the $\bar{c}c$ quarkonia agree with experiment within about 4 MeV i.e. within about 0.1 per cent of the total mass, or one per cent of the first excitation energy. Since the mean square velocity of the c-quark in the ground state of the $\bar{c}c$ system is about 0.25, i.e. the mean root square velocity is about half the velocity of light, this good fit is even more striking than that for the $\bar{b}b$ case. Then we make predictions for the yet undiscovered states hoping that, since our fits for the known states are good, the predictions will also work. One should keep in mind, however, that the problem of the $\bar{b}c$ quarkonia is not just an extrapolation between the $\bar{b}b$ and $\bar{c}c$ cases. If our handling of the effects of the mass difference between the two constituents is faulty, the error of the predictions may be larger than expected.

Secondly, we supplement our nonrelativistic Hamiltonian with the standard spin-dependent terms. With one more parameter – the coupling constant $\alpha_s(m_c^2)$ – we describe all the hyperfine and fine splittings, as well as the

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Table 1. Mass spectrum of the $\bar{b}b$ quarkonia below the threshold for strong decays ($2m_B =$ 10558 MeV) in MeV. ΔX denotes the difference between the mass of particle X and the centre of gravity of the spin triplet part of the multiplet, where X belongs

State	KR [9]	EQ [7]	Present paper	Exp.	State	KR [9]	EQ [7]	Present paper
$\Delta 1\,^1S_0$	–	-87	-56.7	–	$1\,^3D$ (c.o.g.)	10156	10127	10155
$1\,^3S_1$	–	9464	9460	9460	$\Delta 1\,^3D_3$	+4	+3	+7.5
$1\,^3P$ (c.o.g.)	9903	9873	9900	9900	$\Delta 1\,^3D_2$	0	-1	-2.2
$\Delta 1\,^3P_2$	–	+13	+13	+13	$\Delta 1\,^3D_1$	-6	-7	-14
$\Delta 1\,^3P_1$	–	-9	-8.6	-8	$\Delta 1\,^1D_2$	0	0	0
$\Delta 1\,^3P_0$	–	-39	-39	-40	$1\,^3F$ (c.o.g.)	10348	–	10348
$\Delta 1\,^1P_1$	0	0	0	–	$\Delta 1\,^3F_4$	0	–	+5.1
$\Delta 2\,^1S_0$	–	-44	-28	–	$\Delta 1\,^3F_3$	+1	–	-0.9
$2\,^3S_1$	–	10007	10023	10023	$\Delta 1\,^3F_2$	0	–	-7.9
$2\,^3P$ (c.o.g.)	10259	10231	10260	10260	$\Delta 1\,^1F_3$	0	–	0
$\Delta 2\,^3P_2$	–	+11	+9.1	+9	$2\,^3D$ (c.o.g.)	10441	–	10438
$\Delta 2\,^3P_1$	–	-7	-6.0	-5	$\Delta 2\,^3D_3$	+3	–	+6.0
$\Delta 2\,^3P_0$	–	-32	-27	-28	$\Delta 2\,^3D_2$	0	–	-1.8
$\Delta 2\,^1P_1$	0	0	0		$\Delta 2\,^3D_1$	-6	–	-11
$\Delta 3\,^1S_0$	–	-41	-20	–	$\Delta 2\,^1D_2$	0	–	0
$3\,^3S_1$	–	10339	10355	10355	$1\,^3G$ (c.o.g.)	–	–	10508
$3\,^3P$ (c.o.g.)	10520	–	10525	–	$\Delta 1\,^3G_4$	–	–	+3.8
$\Delta 3\,^3P_2$	+6	–	+7.3	–	$\Delta 1\,^3G_3$	–	–	-0.4
$\Delta 3\,^3P_1$	-4	–	-4.9	–	$\Delta 1\,^3G_2$	–	–	-5.4
$\Delta 3\,^3P_0$	-19	–	-22	–	$\Delta 1\,^1G_3$	–	–	0
$\Delta 3\,^1P_1$	0	–	0	–				

leptonic decay widths. The value of $\alpha_s(m_c^2)$ is so plausible – it corresponds to $\alpha_s(m_Z^2) = 0.115$ – that one can either interpret it as a determination of $\alpha_s(m_Z^2)$, or as a known quantity and not a free parameter. We choose the former possibility. By comparison with experiment we find that the errors of the calculated splittings do not exceed about 1 MeV for the $\bar{b}b$ systems and about 5 MeV for the $\bar{c}c$ systems. Also the leptonic decay widths agree reasonably well with experiment. Again by analogy we expect that our predictions for the yet undiscovered states should be good.

2 Spin averaged masses of spin triplet states

The nonrelativistic potential used in the present paper for the colour-singlet $Q\overline{Q}$-systems, where $\overline{Q}$ may, but does not have to, be the charge conjugate of Q, is

$$V(r) = m_{\overline{Q}} + m_Q - 0.78891 + 0.70638\sqrt{r} - 0.32525\frac{1}{r}, \quad (1)$$

where all the constants are in suitable powers of GeV and the quark masses are

$$m_b = 4.8030 \text{ GeV}, \qquad m_c = 1.3959 \text{ GeV} \qquad (2)$$

For the $\bar{b}c$ quarkonia we use the reduced mass

$$\mu_{bc} = \frac{m_b m_c}{m_b + m_c} = 1.0816 \text{ GeV}. \qquad (3)$$

The parameters are given with so many digits only in order to assist the reader, who would like to check our calculations. We do not claim to have established the quark masses with such accuracies. Similar remarks apply to coupling constants etc. given further. For the $\bar{b}b$ system this potential reduces to the potential given in I.

According to our interpretation the eigenvalues of the nonrelativistic Hamiltonian with this potential should be interpreted as the masses of the centres of gravity of the spin triplets. Another popular interpretation is that the nonrelativistic Hamiltonian should give the centres of gravity of the full spin multiplets. Let us consider first the $\bar{b}b$ and $\bar{c}c$ quarkonia. For the $L > 0$ states in our model and in many others the mass of the spin singlet component of each multiplet coincides with the centre of gravity of the spin triplet. Thus the two interpretations are equivalent. The difference occurs only for the S-states. Since the success of the nonrelativistic models depends on a cancellation of errors, which is not understood, we cannot give a rigorous argument to justify one interpretation rather than the other. Let us quote, however, two plausibility arguments. Roncaglia and collaborators [6] explain that the spectrum of triplet states should be particularly regular. We chose the centres of gravity of the triplets for the practical reasons that the masses of the spin singlets for the $\bar{b}b$ quarkonia are not known. *A posteriori*, however, we see that this choice has worked, which supports the conjecture that it is an acceptable one. For the $\bar{b}c$ quarkonia the situation is even more confused, because spin, in general, is not a good quantum number. We assume tentatively that

Table 2. Mass spectrum of the $\bar{c}c$ quarkonia below the threshold for strong decays ($2m_D = 3729$ MeV) in MeV. ΔX denotes the difference between the mass of particle X and the centre of gravity of the spin triplet part of the multiplet, where X belongs

State	EQ [7]	GJ [10]	Present paper	Exp.
$\Delta 1\,^1S_0$	-117	-117	-117	-117
$1\,^3S_1$	3097	3097	3097	3097
$1\,P$ (c.o.g.)	3492	3526	3521	3525
$\Delta 1\,^3P_2$	+15	+31	+31	+31
$\Delta 1\,^3P_1$	-6	-15	-19	-15
$\Delta 1\,^3P_0$	-56	-110	-100	-110
$\Delta 1\,^1P_1$	+1	1	0	+1??
$\Delta 2\,^1S_0$	-78	-68	-72	-92??
$2\,^3S_1$	3686	3685	3690	3686

the eigenvalues of the nonrelativistic Hamiltonian correspond to the centres of gravity of the spin triplets, when the singlet-triplet mixing is switched off. For $L > 0$ the difference between these centres of gravity and the centres of gravity of the whole spin multiplets are small – 3 MeV, 2 MeV and 0.8 MeV for the $1P$, $2P$ and $1D$ multiplets respectively. For the S-states the situation should be similar to that for the other quarkonia.

The results of our calculation for the ten $\bar{b}b$ masses of interest are given in Table 1. In the five cases, where comparison with experiment is possible, agreement is very good. A comparison with some models is also shown in the Table. Eichten and Quigg [7] use the potential of Buchmüller and Tye [8]. This potential is particularly respectable, because at short distances it reproduces correctly the two-loop result from QCD. Their agreement with experiment is very good for the low masses, but for the higher masses it deteriorates, with the error reaching almost 30 MeV for the $2P$ state. Kwong and Rosner [9] use the masses of the known S and P states as input, calculate from this input the potential and from this potential the masses of the other states. Thus their method avoids the bias due to a preconceived form of the potential. Their results for the centres of gravity of the triplets agree in general with ours, the greatest difference being 6 MeV for the $3P$ states.

The results for the $\bar{c}c$ quarkonia are shown in Table 2. It is seen that the approach of Eichten and Quigg is doing much better here, about as well as ours. The results of Gupta and Johnson are the best.

The results for the $\bar{b}c$ quarkonia are given in Table 3. Since there are no experimental data, we included more theoretical predictions. Many others can be found from the references given in the papers used in our tables. Of particular interest is the comparison of our results with those of Gupta and Johnson [10]. Their fit is only slightly worse than ours for the $\bar{b}b$ quarkonia and slightly better than ours for the $\bar{c}c$ quarkonia. On the whole the quality of the two fits is good (errors not exceeding a few MeV) and comparable. Nevertheless, the physical assumptions

Table 3. Mass spectrum of the $\bar{c}b$ ($\bar{b}c$) quarkonia below the threshold for strong decays ($m_D + m_B = 7143$ MeV, $m_{B^*} + m_D = 7189$ MeV) in MeV. ΔX denotes the difference between the mass of particle X and the centre of gravity of the spin triplet part of the multiplet, where X belongs. (a) – particle above its strong decay threshold

State	CK [11]	EQ [7]	Ron [6]	Ger [12]	GJ [10]	Present paper
$\Delta 1\,^1S_0$	-45	-73	-65	-64	-41	-58
$1\,^3S_1$	6355	6337	6320	6317	6308	6349
$1\,P$ (c.o.g.)	6764	6736	6753	6728	6753	6769
$\Delta 1\,^3P_2$	+9	+11	+27	+15	+20	+18
$\Delta 1\,P_1$	0	0	0	+1	+4	+2.4
$\Delta 1\,P_{1'}$	-4	-6	-13	-11	-15	-15
$\Delta 1\,^3P_0$	-36	-36	-93	-45	-64	-54
$\Delta 2\,^1S_0$	-27	-43	–	-35	-33	-33
$2\,^3S_1$	6917	6899	6900	6902	6886	6921
$1\,D$ (c.o.g.)	–	7009	–	7009	–	7040
$\Delta 1\,^3D_3$	–	-4	–	+7	–	+9.3
$\Delta 1\,D_2$	–	0	–	-2	–	0
$\Delta 1\,D_{2'}$	–	+3	–	-8	–	-2.5
$\Delta 1\,^3D_1$	–	+3	–	-1	–	-18
$2\,P$ (c.o.g.)	7160	7142	–	7122	–	7165
$\Delta 2\,^3P_2^{(a)}$	+6	+11	–	+12	–	+13
$\Delta 2\,P_1$	0	0	–	+2	–	+1.9
$\Delta 2\,P_{1'}$	-1	-7	–	-9	–	-11
$\Delta 2\,^3P_0$	-26	-34	–	-34	–	-39

behind them are very different. In particular Gupta and Johnson use the complete relativistic expression for the kinetic energy of the quarks. They also use many more free parameters than we do. For the $\bar{b}c$ quarkonia, as shown in Table 3, the predictions of the two approaches differ significantly. The masses of the ^3S states calculated by Gupta and Johnson are larger than ours by about 40 MeV. For the $1P$ state the difference is in the same direction, but smaller – about 16 MeV. The potential in the model of Yu-Qi Chen and Yu-Ping Kuang [11] is a modification of the potential of Buchmüller and Tye. We quote only one of the several versions, which they propose. The model of Roncaglia et al. [6] does not use a potential, but obtains the masses by assuming that for a given kind of triplet resonances (e.g. for ground states) the mass of the particle is a simple function of the reduced mass of its constituent quarks. Then the masses of the unknown particles are obtained by interpolation or extrapolation from the known masses. Gershtein et al. [12] use the Martin potential [13] supplemented with some relativistic and QCD inspired corrections. The scatter of the predictions is of some tens of MeV and the two models, which agree particularly well with the data for the $\bar{b}b$ and $\bar{c}c$ quarkonia, i.e. that of Gupta and Johnson and ours, are not close to each other in their predictions here. A comparison with the experimental data, when they come, will be, therefore, of great interest.

3 Hyperfine splittings

The spin dependent correction to the nonrelativistic Hamiltonian, which is responsible for the hyperfine splitting of the mass levels, is generally used in the form (cf. e.g. [14])

$$H_{HF} = \frac{32\pi\alpha_s}{9m_Q m_{\overline{Q}}}(\mathbf{s_1}\cdot\mathbf{s_2} - \frac{1}{4})\delta(\mathbf{r}), \quad (4)$$

adapted from the Breit-Fermi Hamiltonian. The number $\frac{1}{4}$ subtracted from the product of the spins corresponds to our assumption that the unperturbed nonrelativistic Hamiltonian gives the energy of the triplet. Since for the states with orbital angular momentum $L > 0$ the wave function vanishes at the origin, the shift affects only the S states. Thus, the only first order effect of the perturbation (4) is to shift the $^1S_0$ states down in energy by

$$\Delta E_{HF} = \frac{32\pi\alpha_s}{9m_Q m_{\overline{Q}}}|\psi(\mathbf{0})|^2. \quad (5)$$

In order to apply this formula one needs the value of the wave function at the origin – this is obtained by solving the Schrödinger equation with the nonrelativistic Hamiltonian – and the coupling constant α_s.

Like most authors (cf. e.g. [7]), we determine the coupling constant $\alpha_s(m_c^2)$ from the well measured hyperfine splitting of the $1S(\bar{c}c)$ state. The experimental value [1] 117 ± 2 MeV yields

$$\alpha_s(m_c^2) = 0.3376. \quad (6)$$

Actually, the experimental uncertainty of the measured hyperfine splitting introduces an uncertainty in this value, but we think that other uncertainties in our calculation are more serious and we do not keep track of this particular uncertainty. Knowing the coupling at the scale m_c^2 we obtain the couplings at other scales as follows. The formula including the NNLO terms from [1] is used to correlate $\alpha_s(\mu^2)$ with the parameters $\Lambda^{(n_f)}$. The number of flavours (n_f) is put equal to three for $\mu^2 \leq m_c^2$ (we are not interested in the region $\mu^2 \leq m_s^2$), equal to four for $m_b^2 \geq \mu^2 \geq m_c^2$ and equal to five for $\mu^2 \geq m_b^2$ (we are not interested in the region $\mu^2 \geq m_t^2$). Then the value of $\alpha_s(m_c^2)$ from (6) is used to calculate $\Lambda^{(3)}$ and $\Lambda^{(4)}$. Using the known value of $\Lambda^{(4)}$ and the formula from [1] we find the value

$$\alpha_s(m_b^2) = 0.2064 \quad (7)$$

From this the value of $\Lambda^{(5)}$ is found and further $\alpha_s(m_Z^2)$ is calculated. The value $\alpha_s(m_Z^2) = 0.115$ obtained from this calculation agrees very well with the other determinations of this parameter compiled by the Particle Data Group [1]. Note that this supports our model, since a different choice of the Hamiltonian would in general lead to a different value of the wave function at the origin and to a different determination of $\alpha_s(m_c^2)$ from the same hyperfine splitting. Then the estimate of $\alpha_s(m_Z^2)$ would, of course, be also different. For the hyperfine splitting of the $\bar{b}c$ quarkonia we use the coupling constant

$$\alpha_s(4\mu_{bc}^2) = 0.2742, \quad (8)$$

so that in each case the scale is twice the reduced mass of the quark-antiquark system.

The calculated hyperfine splittings are given in Tables 1–3. No confirmed experimental data to check these predictions are available as yet. Let us note, however, that the unconfirmed experimental splitting of the $2S(\bar{c}c)$ level – 92 MeV – is much bigger than expected from the models. In all cases, where comparison with the other models is possible, the hyperfine splittings predicted from our model are significantly smaller than the splittings found by Eichten and Quigg [7] and similar to, but usually a little larger than, the splittings calculated by Gupta and Johnson [10].

One can also try to compare our results with more ambitious approaches. A careful analysis in the framework of QCD sum rules [15] finds for the hyperfine splitting of the $1S(\bar{b}b)$ state 63^{+29}_{-51} MeV. The central value agrees very well with our expectation, but the uncertainty is too large to distinguish between the potential models. A lattice calculation [16] gives for the hyperfine splitting of the $1S(\bar{b}c)$ state 60 MeV with a large uncertainty. Again the central value is close to our model, but the uncertainty is big enough to be consistent with all the potential models quoted here.

Let us conclude this section with two comments. The operator H_{HF}, besides shifting the $^1S_0$ energy levels by its diagonal matrix elements, mixes the $^1S_0$ states corresponding to various principal quantum numbers. Formally this gives significant corrections to the energy levels. These corrections, however, are second and higher order in the perturbation. We follow the usage of assuming that they cancel with other second and higher order corrections beyond our control. At first order there are only small admixtures of other $^1S_0$ states in any given $^1S_0$ state. These, however, seem of little interest. The QCD corrections to hyperfine splittings have been calculated in various approximations (cf. [17] and references given there). Since these corrections are small and controversial (it has been argued that they cancel with other correction [17]), we do not include them in our model.

4 Leptonic decay widths

The leading terms in the leptonic decay widths of the heavy quarkonia are proportional to the squares of the wave functions at the origin. Therefore, they are significant only for the S states. For the $\bar{b}b$ quarkonia and the $\bar{c}c$ quarkonia we shall consider the decays of the ^3S (vector) states into pairs of charge conjugated charged leptons, e.g. for definiteness into e^+e^- pairs. For the $\bar{b}c$ quarkonia we consider the decays of the ^1S (pseudoscalar) states into $\tau\nu_\tau$ pairs. Since the probability of such decays contains as a factor the square of the lepton mass, the decays into lighter leptons are much less probable.

Table 4. Leptonic widths in keV

State	EQ [7]	Present paper	Experiment
$1\,^3S_1$ $(\bar{c}c)$	8	4.5 ± 0.5	5.3 ± 0.4
$2\,^3S_1$ $(\bar{c}c)$	3.7	1.9 ± 0.2	2.1 ± 0.2
$1\,^1S_0$ $(\bar{c}b)$	$4.0 \cdot 10^{-8}$	$2.8 \cdot 10^{-8}$	—
$2\,^1S_0$ $(\bar{c}b)$	—	$1.6 \cdot 10^{-8}$	—
$1\,^3S_1$ $(\bar{b}b)$	1.7	1.36 ± 0.07	1.32 ± 0.05
$2\,^3S_1$ $(\bar{b}b)$	0.8	0.59 ± 0.03	0.52 ± 0.03
$3\,^3S_1$ $(\bar{b}b)$	0.6	0.40 ± 0.02	0.48 ± 0.08

The decay widths of the vector $\bar{b}b$ and $\bar{c}c$ quarkonia into charged lepton pairs are usually calculated from the QCD corrected Van Royen - Weisskopf formula [18], [19]

$$\Gamma_{V\to\bar{l}l} = 16\pi\alpha^2 e_Q^2 \frac{|\psi(\mathbf{0})|^2}{M_V^2}\left(1 - \frac{16\alpha_s(m_Q^2)}{3\pi}\right). \quad (9)$$

For vector mesons containing light quarks this formula leads to paradoxes (cf. [20] and references contained there). For quarkonia, however, the main problem seems to be the QCD correction. Using the coupling constants $\alpha_s(m_Q^2)$ found in the preceding section one finds that the correction linear in α_s is 57 per cent for $\bar{c}c$ and 35 per cent for $\bar{b}b$. Thus in order to get quantitative predictions it is necessary to include higher order corrections, which, however, are not known. In order to guestimate the missing terms we tried two simple Ansätze. Exponentialization of the first correction yields

$$C_1(\alpha_s(m_Q^2)) = \exp\left(-\frac{16\alpha_s(m_Q^2)}{3\pi}\right), \quad (10)$$

while Padéization gives

$$C_2(\alpha_s(m_Q^2)) = \frac{1}{1 + \frac{16\alpha_s(m_Q^2)}{3\pi}}. \quad (11)$$

We use the arithmetic average of these two estimates as our estimate of the QCD correction factor extended to higher orders. The difference between C_1 and C_2 is our crude evaluation of the uncertainty of this estimate. The resulting leptonic widths are collected in Table 4. Combining in quadrature the experimental errors with our estimates of the theoretical uncertainties we get a good overall agreement ($\chi^2/ND = 5.9/5$). About half of the χ^2, however, comes from the decay width of the $2^3S(\bar{b}b)$, where the predicted value is significantly larger than the newly included experimental value [1]. Thus here there may be a problem. Let us note the relation

$$\Gamma_{V\to\bar{l}l} = \frac{9}{8}\,\frac{4m_Q^2}{M_V^2}\,\frac{\alpha^2 e_Q^2}{\alpha_s(m_Q^2)} C_{av}\Delta E_{HF}, \quad (12)$$

where C_{av} is the QCD correction factor. With our choice of parameters this formula reduces to

$$\Gamma_{V\to\bar{l}l} = F(Q)\frac{4m_Q^2}{M_V^2}\Delta E_{HF}, \quad (13)$$

with $F(c) = 4.73 \cdot 10^{-5}$ and $F(b) = 2.33 \cdot 10^{-5}$.

The formula for the leptonic widths of the pseudoscalar $\bar{b}c$ quarkonia reads

$$\Gamma_{\tau\nu} = \frac{G^2}{8\pi} f_{B_c}^2 |V_{cb}|^2 M_{B_c} m_\tau^2 \left(1 - \frac{m_\tau^2}{M_{B_c}^2}\right)^2, \quad (14)$$

where G is the Fermi constant, $V_{cb} \approx 0.04$ is the element of the Cabibbo-Kobayashi-Masakawa matrix and the decay constant f_{B_c} is given by the formula (cf. e.g. [21])

$$f_{B_c}^2 = \frac{12|\psi(\mathbf{0})|^2}{M_{B_c}}\overline{C}^2(\alpha_s), \quad (15)$$

where $\overline{C}(\alpha_s)$ is a QCD correction factor. Formally this decay constant is defined in terms of the element of the axial weak current

$$\langle 0|A_\mu(0)|B_c(q)\rangle = i f_{B_c} V_{cb} q_\mu. \quad (16)$$

Thus it corresponds to $f_\pi \approx 131$ MeV. The QCD correction factor is [21]

$$\overline{C}(\alpha_s) = 1 - \frac{\alpha(4\mu_{bc}^2)}{\pi}\left[2 - \frac{m_b - m_c}{m_b + m_c}\log\frac{m_b}{m_c}\right]. \quad (17)$$

With our parameters $\overline{C}(\alpha_s) \approx 0.885$ and since this is rather close to unity, we use it without trying to estimate the higher order terms.

Substituting the numbers one finds the decay widths given in Table 4. The corresponding decay constants for the ground state and for the first excited S-state of the $\bar{b}c$ quarkonium are $f_{B_c} = 435 MeV$ and $f_{B_c} = 315$ MeV.

Let us note the convenient relation

$$f_{B_c}^2 = \frac{27\mu_{bc}}{8\pi\alpha_s(4\mu_{bc}^2)}\frac{m_b + m_c}{M_{B_c}}\overline{C}^2(\alpha_s)\Delta E_{HF}, \quad (18)$$

which for our values of the parameters yields

$$f_{B_c} = 57.6\sqrt{\frac{6199}{M_{B_c}}}\sqrt{\Delta E_{HF}}, \quad (19)$$

where all the parameters are in suitable powers of MeV.

5 Fine structure of the levels

The spin dependent correction to the nonrelativistic Hamiltonian, which is responsible for the fine splittings, is also modelled on the Breit-Fermi Hamiltonian [7], [12]. It can be decomposed into a part, which is antisymmetric with respect to the spins of the constituents

$$V_A(r) = \frac{1}{4}\left(\frac{1}{m_Q^2} - \frac{1}{m_{\overline{Q}}^2}\right)\left(-\frac{1}{r}\frac{dV(r)}{dr} + \frac{8\alpha_s}{3r^3}\right)\mathbf{L} \cdot(\mathbf{s}_Q - \mathbf{s}_{\overline{Q}}), \quad (20)$$

where $V(r)$ is the nonrelativistic potential (1), and a part symmetric in these spins. The symmetric part can be decomposed into a spin-orbit coupling

$$V_{LS}(r) = \frac{1}{4}\left(\frac{1}{m_Q^2} + \frac{1}{m_{\overline{Q}}^2}\right)\left(-\frac{1}{r}\frac{dV(r)}{dr} + \frac{8\alpha_s}{3r^3}\right)\mathbf{L}\cdot\mathbf{S} + \frac{4\alpha_s}{3m_Q m_{\overline{Q}}}\frac{1}{r^3}\mathbf{L}\cdot\mathbf{S}, \quad (21)$$

where

$$\mathbf{S} = \mathbf{s}_Q + \mathbf{s}_{\overline{Q}}, \quad (22)$$

and a tensor part

$$V_T(r) = \frac{4\alpha_s}{3}\frac{1}{r^3}[3(\mathbf{s}_Q\cdot\hat{\mathbf{n}})(\mathbf{s}_{\overline{Q}}\cdot\hat{\mathbf{n}}) - \mathbf{s}_Q\cdot\mathbf{s}_{\overline{Q}}], \quad (23)$$

where the versor $\hat{\mathbf{n}} = \frac{\mathbf{r}}{r}$. In the first perturbative approximation these corrections to the Hamiltonian not only shift the mass levels, but also mix some states with different values of the orbital angular momentum and spin.

Let us note some useful selection rules. Angular momentum and parity are good quantum numbers, therefore states with different J and/or P do not mix. States with orbital angular momenta differing by one unit do not mix, because parity is a good quantum number, and states with angular momenta differing by more than two units cannot mix because of the Eckart-Wigner theorem. From symmetry with respect to the exchange of spins in the LS basis, the operator V_A can only contribute to matrix elements between the spin singlet and spin triplet states. The other two operators contribute only to the matrix elements between spin triplet states. When $\overline{Q}$ is the charge conjugate of Q, charge conjugation $C = (-1)^{L+S}$ is a good quantum number and states with different spins do not mix. The matrix elements between spin singlet states vanish; thus, when spin is a good quantum number, we predict within each spin multiplet with $L > 0$:

$$M(^1L) = M_{c.o.g.}(^3L), \quad (24)$$

Here the centre of gravity of the triplet is defined by the usual formula

$$M_{c.o.g.}(^3L) = \left[(2L+3)M(^3L_{L+1}) + (2L+1)M(^3L_L) + (2L-1)M(^3L_{L-1})\right]\big/\left[3(2L+1)\right]. \quad (25)$$

There is no firmly established experimental data to test this prediction, but most models and the preliminary data for the $1^1P_1(\bar{c}c)$ state agree with it within about 1 MeV. Of course, this prediction is common to all the models, which use the spin dependent corrections to the Hamiltonian as given here.

Since the wave function of the quarkonium in the LS basis factorizes into a space part and a spin part, the matrix elements of the space operators and of the spin operators can be calculated separately. The necessary matrix elements for the spin operators between spin triplet states are

$$\langle J, L', 1|\mathbf{L}\cdot\mathbf{S}|J, L, 1\rangle = \frac{1}{2}[J(J+1) - L(L+1) - 2]\delta_{LL'} \quad (26)$$

$$\langle L+1, L, 1|3(\mathbf{s}_Q\cdot\hat{\mathbf{n}})(\mathbf{s}_{\overline{Q}}\cdot\hat{\mathbf{n}}) - \mathbf{s}_Q\cdot\mathbf{s}_{\overline{Q}}|L+1, L, 1\rangle = -\frac{2L}{2L+3}, \quad (27)$$

$$\langle L, L, 1|3(\mathbf{s}_Q\cdot\hat{\mathbf{n}})(\mathbf{s}_{\overline{Q}}\cdot\hat{\mathbf{n}}) - \mathbf{s}_Q\cdot\mathbf{s}_{\overline{Q}}|L, L, 1\rangle = 2, \quad (28)$$

$$\langle L-1, L, 1|3(\mathbf{s}_Q\cdot\hat{\mathbf{n}})(\mathbf{s}_{\overline{Q}}\cdot\hat{\mathbf{n}}) - \mathbf{s}_Q\cdot\mathbf{s}_{\overline{Q}}|L-1, L, 1\rangle = -\frac{2(L+1)}{2L-1}, \quad (29)$$

$$\langle L+1, L-2, 1|3(\mathbf{s}_Q\cdot\hat{\mathbf{n}})(\mathbf{s}_{\overline{Q}}\cdot\hat{\mathbf{n}}) - \mathbf{s}_Q\cdot\mathbf{s}_{\overline{Q}}|L+1, L, 1\rangle = \frac{3\sqrt{L(L-1)}}{2(2L-1)}. \quad (30)$$

The only necessary nonzero matrix element between a spin singlet and spin triplet state is

$$\langle L, L, 0|\mathbf{L}\cdot(\mathbf{s_Q} - \mathbf{s_{\overline{Q}}})|L, L, 1\rangle = \sqrt{L(L+1)} \quad (31)$$

Let us consider now the two matrix elements in coordinate space. The calculation of the matrix elements of the operator $\frac{1}{r}\frac{dV(r)}{dr}$ is standard, but the calculation of the matrix elements of the operator $\frac{\alpha_s}{r^3}$ requires an interpretation of α_s. We propose to interpret α_s as a function $\tilde{\alpha}_s(r)$ defined as follows

$$\tilde{\alpha}_s(r) = \frac{12\pi}{33 - 2n_f}\frac{(\tilde{\Lambda}^{(n_f)}r)^2 - 1}{\log\left[(\tilde{\Lambda}^{(n_f)}r)^2\right]}, \quad (32)$$

where n_f equals three for $r < \frac{1}{m_c}$, equals four for $\frac{1}{m_c} < r < \frac{1}{m_b}$ and equals five for $r > \frac{1}{m_b}$. The parameter $\tilde{\Lambda}^{(4)}$ is obtained from the conditions $\tilde{\alpha}(\frac{1}{m_c}) = \alpha(m_c^2)$ and $\tilde{\alpha}(\frac{1}{m_b}) = \alpha(m_b^2)$. Each of these conditions gives a slightly different $\tilde{\Lambda}^{(4)}$. We use the geometric mean of the two results. They are so close to each other that taking the arithmetic mean instead of the geometrical one makes no difference within our precision. Knowing $\tilde{\Lambda}^{(4)}$ we recalculate $\tilde{\alpha}_s(r)$ at $r = \frac{1}{m_b}$ and $r = \frac{1}{m_c}$ and fix $\tilde{\Lambda}^{(3)}$ and $\tilde{\Lambda}^{(5)}$ so that the function $\tilde{\alpha}_s(r)$ is continuous at these points. We find

$$\tilde{\Lambda}^{(3)} = 0.1657 \text{ GeV}, \quad \tilde{\Lambda}^{(4)} = 0.1384 \text{ GeV}, \quad \tilde{\Lambda}^{(5)} = 0.1015 \text{ GeV}. \quad (33)$$

Our form of the function $\tilde{\alpha}_s(r)$ is, of course, inspired by the standard one-loop formula for α_s. The numerator is introduced in order to compensate the zero of the denominator at $\tilde{\Lambda}^{(3)}r = 1$. Its exact form has little effect in the range of r dominating the integrals.

The calculations for the $\bar{b}b$ and $\bar{c}c$ quarkonia, where C is a good quantum number, and for the $J = L\pm1$ states of the $\bar{b}c$ quarkonia, which must be spin triplets, involve only the energy shifts due to the symmetric spin orbit interaction and to the tensor interaction. The singlet and triplet

$J = L$ states of the $\bar{b}c$ quarkonia mix under the influence of the antisymmetric spin-orbit interaction. The results of the calculations are given in Tables 1-3. For the $\bar{b}b$ quarkonia experimental data is available for the splittings of the $1P$ and $2P$ states. Agreement between our model and this data is within about 1 MeV. The agreement with other models is within 5 MeV, i.e. much better than for the hyperfine splittings. A similar agreement with the model of Kwong and Rosner holds for the $3P$ states, but for higher angular momenta the discrepancies increase. We predict much larger splittings. In particular for the F states we predict a splitting of about 13 MeV, while Kwong and Rosner expect a negligible splitting within about 1 MeV. Even for the G states we expect a splitting of about 10 MeV. Thus, the L-dependence of the fine splittings is seen as an important observable to distinguish between models. For the $1P(\bar{c}c)$ states our splittings agree with experiment and with the very good predictions of Gupta and Johnson about as well as for the centres of gravity of the triplets i.e. within about 5 MeV. The splittings predicted by the model of Eichten and Quigg are too small by about a factor of two. For the $\bar{b}c$ quarkonia our predictions for the $1P$ states agree with Gupta and Johnson within about 2 MeV except for $J = 0$, where our splitting is smaller by 10 MeV. For the $1D$ and $2P$ states there are only the predictions of Eichten and Quigg [7] and of Gershtein et al. [12] for comparison. There is rough qualitative agreement except for the $^3D_1$ state, where we predict a down shift by almost 20 MeV, while the other models find only very small shifts.

The mixing between the spin singlet and the spin triplet states can be parameterized in terms of mixing angles. We find $\sin\phi_{1P} = 0.374$, $\sin\phi_{2P} = 0.385$ and $\sin\phi_{1D} = 0.244$. Thus the mixing within the two P multiplets is almost the same, while the mixing among the D states is somewhat smaller. Both these results contradict the results of Gershtein and collaborators [12], who find that mixing increases when going from the $1P$ to the $2P$ states and from the $2P$ to the $1D$ states. A possible reason for this discrepancy is that these authors use for the mixing formulae, which are different from ours. In particular their singlet-triplet mixing does not vanish for $m_Q = m_{\overline{Q}}$. As compared with Eichten and Quigg, who have calculated mixing only for the P states, we have rough agreement for the $2P$ states (they find $\sin\phi_{2P} = 0.290$), while they find much less mixing for the $1P$ states ($\sin\phi_{1P} \approx 0.06$).

The mixing of spin triplet states differing by two units of orbital angular momentum $(L-2,\ L)$ is small. In our model we find mixing angles of order 10^{-3} or less. It seems of little interest, except that it enhances the leptonic decay widths of the $L \geq 2$ states (cf. e.g. [22]). This enhancement, however, is difficult to calculate reliably, because a given high L state mixes with various $L-2$ states and the states above the strong decay threshold are also important for this analysis.

6 Conclusions

We propose a model containing six free parameters: the three parameters in the nonrelativistic potential (1), the masses of the c and b quarks (2) and the strong coupling at the m_c scale (6). This model is applicable to all the heavy quarkonia below their strong decay thresholds.

We obtain for the $\bar{b}b$ quarkonia 12 quantities (five spin averaged masses, four independent mass differences due to fine splittings and three leptonic decay widths) in good overall agreement with experiment. The least successful predictions are for the fine structure shift of the 2^3P_1 state, which is measured to be -4.8 ± 0.5 MeV, while we find -6.0 MeV and for the leptonic decay width of the $2^3S(\bar{b}b)$ state, where we find (0.59 ± 0.03) KeV, while the newly included experimental result [1] is (0.52 ± 0.03) KeV. For the $\bar{c}c$ quarkonia we find 8 quantities, which can be compared with experiment (six masses and two leptonic widths). Here in most cases the difference between the measured value and the prediction exceeds the experimental error, but the errors in the mass predictions do not exceed a few MeV and the errors in the leptonic decay widths do not exceed 1.6 s.d.. On the whole, with 6 parameters we predict 20 quantities in good (for $\bar{b}b$) or fair (for $\bar{c}c$) agreement with experiment. The only other model known to us, which has a comparable record, is the model of Gupta and Johnson [10], but this model has many more free parameters.

We give predictions for the yet unmeasured masses of the quarkonia and for the leptonic widths of the $\bar{b}c$ quarkonia. The predictions are listed in Tables 1-4. Here we would like to make the following general remarks. Our model predicts much larger fine splittings at high L than the model of Kwong and Rosner [9]. We also find significantly heavier $\bar{b}c$ resonances than Gupta and Johnson [10], which is remarkable, because the two models give similar descriptions of the charmonia and of the bottomonia. The discrepancy is in the spin averaged masses. The splittings of the levels, except for the hyperfine splitting of the $1S$ level, are similar.

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